数理解析研究所講究録
503巻 1983年 61-71

A description of a space of holomorphic discrete series by means of the Fourier transform on the Šilov boundary

京大理 野村隆昭 (Takaaki Nomura)

1. Let G be a connected non-compact real simple Lie group of matrices and K a maximal compact subgroup of G. Assume that G/K is a hermitian symmetric space. Then, G/K can be realized as a Siegel domain of type II, D, in $W \times V$, where W and V are finite dimensional complex vector spaces. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in the Lie algebra $\mathfrak{k}$ of K, Δ the root system of $(\mathfrak{g}_C, \mathfrak{h}_C)$. We introduce an order in Δ compatible with the complex structure of G/K. For each K-dominant K-integral linear form Λ on $\mathfrak{h}_C$ satisfying the Harish-Chandra's non-vanishing condition [1], the holomorphic discrete series Π_Λ of G with Λ as a parameter is realized on a Hilbert space $\mathcal{H}(\Lambda)$(†) of vector valued holomorphic functions on D. Let $S(D)$ be the Šilov boundary of D. Then, one knows that $S(D)$ is the orbit of the origin in $W \times V$ under a certain nilpotent subgroup $N(D)$ of the affine automorphisms of D, and that $S(D)$ is diffeomorphic to $N(D)$. By identifying $S(D)$ with $N(D)$, the aim of this talk is a description of the space $\mathcal{H}(\Lambda)$ by using the Fourier transform on $N(D)$.

If D reduces to a tube domain, then $N(D)$ is abelian and therefore the Fourier transform on $N(D)$ is euclidean. Since this case is treated

(†) For precise definition see p.7.

in Rossi-Vergne [4], we assume from now on that D does not reduce to a tube domain. Then, $N(D)$ is a simply connected 2-step nilpotent Lie group (in fact $N(D)$ is the nilradical of a maximal parabolic subgroup of G) and the Fourier transform on $N(D)$ is non-euclidean.

2. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ with the associated Cartan involution θ and $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) be the sum of all the root subspaces corresponding to positive (resp. negative) non-compact roots in Δ. Both $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are abelian subalgebras in $\mathfrak{g}_C$ normalized by $\mathfrak{k}_C$. Let $P_\pm$ and K_C be the analytic subgroups of G_C corresponding to $\mathfrak{p}_\pm$ and $\mathfrak{k}_C$ respectively, where G_C is the complexified connected matrix group having $\mathfrak{g}_C$ as Lie algebra. Every $x \in P_+K_CP_-$ can be expressed in a unique way as

$$x = \exp \zeta(x) \cdot k(x) \cdot \exp \zeta'(x)$$

with $\zeta(x) \in \mathfrak{p}_+$, $k(x) \in K_C$ and $\zeta'(x) \in \mathfrak{p}_-$. We know that G is contained in $P_+K_CP_-$ and that the image $M = \zeta(G)$ is a bounded domain in $\mathfrak{p}_+$. M is known as the Harish-Chandra's realization of G/K.

3. Let $\{\gamma_1, \ldots, \gamma_\ell\}$ be a maximal system of positive non-compact pairwise strongly orthogonal roots constructed as follows: for each j, γ_j is the largest positive non-compact root strongly orthogonal to $\gamma_{j+1}, \ldots, \gamma_\ell$. For every $\alpha \in \Delta$, we choose $H_\alpha \in \mathfrak{h}_C$ and $X_\alpha \in \mathfrak{g}_\alpha \subset \mathfrak{g}_C$ in such a way that the following are valid:

$$B(H_\alpha, H) = \alpha(H) \quad (H \in \mathfrak{h}_C), \qquad X_\alpha - X_{-\alpha} \in \mathfrak{k} + i\mathfrak{p},$$
$$i(X_\alpha + X_{-\alpha}) \in \mathfrak{k} + i\mathfrak{p}, \qquad [X_\alpha, X_{-\alpha}] = 2H_\alpha/\alpha(H_\alpha),$$

where B is the Killing form of $\mathfrak{g}_C$. Note that $H_\alpha \in i\mathfrak{h}$. Put $\mathfrak{a} = \sum_{1 \leqq i \leqq \ell} R(X_{\gamma_i} + X_{-\gamma_i})$. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$, so that ℓ equals the real rank of G. Let

$$(1) \qquad c = \exp \frac{\pi}{4} \sum_{j=1}^{\ell} (X_{\gamma_j} - X_{-\gamma_j}) \in G_C$$

and $\nu = \mathrm{Ad}\, c$. It should be noted that c belongs to $P_+K_CP_-$. Put $\mathfrak{h}^- = \sum_{1 \leqq j \leqq \ell} RH_{\gamma_j}$ and let $\mathfrak{h}^+$ be the orthogonal complement of $\mathfrak{h}^-$ in $i\mathfrak{h}$ relative to the inner product $-B_\tau\big|_{i\mathfrak{h}\times i\mathfrak{h}}$, where $B_\tau(X, Y) = B(X, \tau Y)$ and τ is the conjugation in $\mathfrak{g}_C$ with respect to the compact real form $\mathfrak{k} + i\mathfrak{p}$. Since $\mathfrak{h}_C^+ + \mathfrak{a}_C = \nu^{-1}(\mathfrak{h}_C)$, $\mathfrak{h}_C^+ + \mathfrak{a}_C$ is also a Cartan subalgebra of $\mathfrak{g}_C$. As we have assumed that G/K does not reduce to a tube domain, there is only one possibility of the positive $\mathfrak{a}$-root system $\Phi(\mathfrak{a})^+$ compatible with the original order in Δ through ν^* (see Moore [2]): put $\lambda_j = \nu^*(\gamma_j)$, then

$$\Phi(\mathfrak{a})^+ = \{\frac{\lambda_i + \lambda_j}{2}\ ;\ 1 \leqq j \leqq i \leqq \ell\} \bigcup \{\frac{\lambda_i - \lambda_j}{2}\ ;\ 1 \leqq j < i \leqq \ell\} \bigcup \{\frac{\lambda_i}{2}\ ;\ 1 \leqq i \leqq \ell\}.$$

We denote by $\mathfrak{n}$ the sum of all the positive $\mathfrak{a}$-root subspaces and put $\mathfrak{s} = \mathfrak{a} + \mathfrak{n}$. It is easy to see that the map $r(X) = (X - \theta(X))/2$ is a linear isomorphism of $\mathfrak{s}$ onto $\mathfrak{p}$. Let j be the complex structure on the underlying vector space $\mathfrak{s}$ obtained by transforming the complex structure on $\mathfrak{p}$ by means of r. We set

$$\mathfrak{s}(0) = \mathfrak{a} + \sum_{k<m} \mathfrak{n}_{(\lambda_m - \lambda_k)/2}, \quad \mathfrak{s}(1/2) = \sum_{k=1}^{\ell} \mathfrak{n}_{\lambda_k/2}, \quad \mathfrak{s}(1) = \sum_{k \leqq m} \mathfrak{n}_{(\lambda_m + \lambda_k)/2}.$$

Then, clearly we have $\mathfrak{s} = \mathfrak{s}(0) + \mathfrak{s}(1/2) + \mathfrak{s}(1)$ and $\mathfrak{s}(0)$ is a subalgebra of $\mathfrak{g}$. Let $S(0)$ be the analytic subgroup of G corresponding to $\mathfrak{s}(0)$.

Put

$$s = \frac{i}{2} \sum_{k=1}^{\ell} (2H_{\gamma_k}/\langle\gamma_k, \gamma_k\rangle - X_{\gamma_k} + X_{-\gamma_k}).$$

Then, s belongs to $\mathfrak{s}(1)$. Let Ω be the $S(0)$-orbit of s in $\mathfrak{s}(1)$ under the adjoint representation. By Rossi-Vergne [4, Theorem 4.15], Ω is a regular open convex cone in $\mathfrak{s}(1)$ and diffeomorphic to $S(0)$. For every $t \in \Omega$, we denote by $\eta_0(t)$ the unique element in $S(0)$ for which $(\mathrm{Ad}\,\eta_0(t))s = t$. On the other hand, it is known that j leaves $\mathfrak{s}(1/2)$ invariant, and so $\mathfrak{s}(1/2)$ can be considered as a complex vector space V by $j|_{\mathfrak{s}(1/2)}$. Let W be the complexification of $\mathfrak{s}(1)$. The $\mathbb{R}$-bilinear map $Q : V \times V \to W$ defined by $Q(x, y) = ([jx, y] + i[x, y])/4$ turns out to be an Ω-positive sesqui-linear hermitian map. By using this pair of Ω and Q, we now define a Siegel domain of type II, $D = D(\Omega, Q)$:

$$D = \{(w, v) \in W \times V \,;\, \mathrm{Im}\, w - Q(v, v) \in \Omega\}.$$

Then, $S(D) = \{(x + iQ(\zeta, \zeta), \zeta) \,;\, x \in \mathfrak{s}(1), \zeta \in V\}$ and $N(D) = \{(x, \zeta) \,;\, x \in \mathfrak{s}(1), \zeta \in V\}$ with the multiplication

$$(x, \zeta)\cdot(x', \zeta') = (x + x' + 2\mathrm{Im}\, Q(\zeta, \zeta'), \zeta + \zeta').$$

Let $\alpha : G \to D$ be the map which induces a G-equivariant biholomorphism of G/K onto D. Then it holds that

$$D \ni (x + iy, \zeta) = \alpha(\exp(x, \zeta)\cdot\eta_0(y - Q(\zeta, \zeta))).$$

4. Let Ξ (resp. Ξ^+) be the set of all $\lambda \in \mathfrak{s}(1)^*$ such that the hermitian form $\langle\lambda, Q(\cdot, \cdot)\rangle$ on $V \times V$ is non-degenerate (resp. positive definite). It can be proved that Ξ^+ is equal to the dual cone

$$\Omega^* = \{\lambda \in \mathfrak{s}(1)^* : \langle\lambda, x\rangle > 0 \quad \text{for all} \quad x \in \overline{\Omega} - \{0\}\}.$$

In particular, Ξ is non-empty. As in Ogden-Vági [3], we have a family $(\pi_\lambda)_{\lambda\in\Xi}$ of concrete irreducible unitary representations of $N(D)$ enough to decompose $L^2(N(D))$. The space of π_λ is the L^2-space on a real subspace E_λ of V with $\dim_{\mathbf{R}} E_\lambda = \dim_{\mathbf{C}} V$ (= n, say). For $\lambda \in \Xi$, let $\rho(\lambda)$ be the Pfaffian of the alternating bilinear form $\mathrm{Im}\langle\lambda, Q(\cdot, \cdot)\rangle$ on the real vector space $\mathfrak{s}(1/2) \times \mathfrak{s}(1/2)$. The Fourier transform $\hat{f}$ of $f \in L^1(N(D))$ is by definition

$$\hat{f}(\lambda) = \int_{N(D)} f(n)\pi_\lambda(n^{-1})\, dn ,$$

where dn is the Haar measure on $N(D)$. Then, the Plancherel formula for $N(D)$ is as follows:

$$||f||^2 = c \int_\Xi ||\hat{f}(\lambda)||_{HS}^2 \, \rho(\lambda)\, d\lambda.$$

The positive constant c depends only on the normalization of dn. One can define the Fourier transform of $f \in L^2(N(D))$ in the standard way.

5. Let ψ be a continuous everywhere positive function on Ω such that $\psi(at) = a^\delta\psi(t)$ $(a > 0,\ t \in \Omega)$ for some $\delta \in \mathbf{R}$. We consider first the Hilbert space $H^2(D, \psi)$ of $\mathbf{C}$-valued holomorphic functions on D satisfying

$$\int_D |F(x + iy, \zeta)|^2 \, \psi(y - Q(\zeta, \zeta)) \, dxdyd\zeta < \infty \qquad (x + iy \in W, \ \zeta \in V).$$

When the irreducible unitary representation of K with highest weight Λ is one dimensional, $H(\Lambda)$ is of this type for a certain ψ. For $F \in H^2(D, \psi)$, we put

$$\text{(2)} \qquad f_t(x, \zeta) = F(x + i(t + Q(\zeta, \zeta)), \zeta)$$

for every $t \in \Omega$. Then, f_t belongs to $L^2(N(D))$, so that one can consider the Fourier transform $\hat{(f_t)}$ of f_t. Identify $L^2(E_\lambda)$ with $L^2(R^n)$ and let $\{\phi_m^\lambda \ ; \ m \in (Z_+)^n\}$ be the complete orthonormal system consisting of the Hermite functions and V_λ the one dimensional subspace of $L^2(E_\lambda)$ spanned by ϕ_0^λ. We now define a Hilbert space $H^2(\Omega^*, \psi)$: every $\Phi \in H^2(\Omega^*, \psi)$ is a measurable function on Ξ taking its value at $\lambda \in \Xi$ in the Hilbert space of Hilbert-Schmidt operators on $L^2(E_\lambda)$ such that

(i) $\Phi(\lambda) = 0$ if $\lambda \notin \Omega^*$,

(ii) Range $\Phi(\lambda)$ is contained in V_λ if $\lambda \in \Omega^*$,

(iii) $||\Phi||^2 = c \int_{\Omega^*} ||\Phi(\lambda)||_{HS}^2 \, I_\psi(\lambda)\rho(\lambda) \, d\lambda < \infty$,

where $I_\psi(\lambda) = \int_\Omega e^{-2\langle\lambda, x\rangle} \psi(x) \, dx$.

Theorem 1. Let $F \in H^2(D, \psi)$ be given and define f_t by (2). Then, $\Phi(\lambda) = e^{\langle\lambda, t\rangle}\hat{(f_t)}(\lambda)$ is independent of $t \in \Omega$ and belongs to $H^2(\Omega^*, \psi)$. Conversely, let $\Phi \in H^2(\Omega^*, \psi)$ be given. Then,

$$F(x + i(t + Q(\zeta, \zeta)), \zeta) = c \int_{\Omega^*} e^{-\langle\lambda, t\rangle} \operatorname{Tr}[\pi_\lambda(x, \zeta)\Phi(\lambda)] \rho(\lambda)\, d\lambda$$

is absolutely convergent and gives an element $F \in H^2(D, \psi)$ such that $\Phi(\lambda) = e^{\langle\lambda, t\rangle} (f_t)^\wedge (\lambda)$. Moreover, the map $F \to \Phi$ is unitary.

6. Now we treat $\mathcal{H}(\Lambda)$. Let Λ be as in 1 and τ_Λ the irreducible unitary representation of K on a finite dimensional Hilbert space E with highest weight Λ. Since P_+K_C is a semidirect product, τ_Λ can be naturally extended to a representation of P_+K_C. Let $c \in G_C$ be the element defined by (1) and put

$$\Phi_\Lambda(g) = \tau_\Lambda(k(c)^{-1})\tau_\Lambda(k(cg)).$$

We note that it makes sense to write $k(cg)$ for $g \in G$, for one can show that $cg \in P_+K_CP_-$. Let

$$\theta_0(t) = |\det_{\mathfrak{s}(1/2)} \operatorname{Ad} \eta_0(t)|^{-1} |\det_{\mathfrak{s}(1)} \operatorname{Ad} \eta_0(t)|^{-2} \qquad (t \in \Omega)$$

and $\Theta_\Lambda(\alpha(h)) = \Phi_\Lambda(h)$ $(h \in S = \exp \mathfrak{s})$, where α is the map $G \to D$ which induces a G-equivariant biholomorphism of G/K onto D.

The Hilbert space $\mathcal{H}(\Lambda)$ consists of E-valued holomorphic functions on D satisfying

$$\|F\|^2 = \int_D \|\Theta_\Lambda(iy, \zeta)^{-1}F(x + iy, \zeta)\|^2\, \theta_0(y - Q(\zeta, \zeta))\, dxdyd\zeta < \infty.$$

Let v_Λ be a highest weight vector for τ_Λ normalized so that $\|v_\Lambda\|$

$= 1$. We take an orthonormal basis $v_1 = v_\Lambda, v_2, \ldots, v_d$ $(d = \deg \tau_\Lambda)$ in E consisting of weight vectors arranged in order so that any vector in the root subspaces corresponding to positive compact roots in Δ is represented, under τ_Λ, by an upper triangular matrix. We denote by Λ_j the weight for the weight vector v_j. Let E_k be the one dimensional subspace of E spanned by v_k and

$$H_j(\Lambda) = \{F \in H(\Lambda)\ ;\ F(w, \zeta) \in E_1 \oplus \cdots \oplus E_j\}.$$

Then, it can be proved that $H_j(\Lambda)$ is a closed subspace of $H(\Lambda)$ invariant under $\Pi_\Lambda|_S$. Let $H^1(\Lambda) = H_1(\Lambda)$ and $H^j(\Lambda) =$ the orthogonal complement of $H_{j-1}(\Lambda)$ in $H_j(\Lambda)$ $(j = 2, \ldots, d)$. Define a positive character χ_j $(j = 1, \ldots, d)$ of $A = \exp \mathfrak{a}$ by

$$\chi_j\left(\exp \sum_{i=1}^{\ell} a_i (X_{\gamma_i} + X_{-\gamma_i})\right) = \prod_{i=1}^{\ell} \exp a_i \Lambda_j(\nu(X_{\gamma_i} + X_{-\gamma_i})),$$

where $\nu = \mathrm{Ad}\, c$ as in 3. Extending χ_j canonically to a character of S, we put

$$\psi_j(t) = \chi_j(\eta_0(t))^{-2}\, \theta_0(t) \qquad (t \in \Omega,\ j = 1, 2, \ldots, d).$$

Then, $\psi_j(at) = a^{\delta_j} \psi_j(t)$ $(a > 0,\ t \in \Omega)$ for some $\delta_j \in R$. Consider the Hilbert space $H^2(D, \psi_j)$ of the type in 5 and define an operator T_j by

$$T_j F(w, \zeta) = (F(w, \zeta), v_j) \qquad (F \in H_j(\Lambda)).$$

T_j is a bounded operator $H_j(\Lambda) \to H^2(D, \psi_j)$ and its range is dense.

Therefore $H^j(\Lambda)$ is unitarily isomorphic to $H^2(D, \psi_j)$ by U_j, where U_j is the partial isometry appearing in the polar decomposition of the operator T_j. Thus, we have an irreducible decomposition $H(\Lambda) = \bigoplus_{j=1}^{d} H^j(\Lambda)$ for $\Pi_\Lambda|_S$.

7. Put for $\lambda \in \Omega^*$

(3) $$I_\Lambda(\lambda) = \int_\Omega e^{-2\langle\lambda,\ t\rangle}\ \Phi_\Lambda(\eta_0(t)^{-1})^2\ \theta_0(t)\ dt.$$

Lemma. The integral in (3) is absolutely convergent.

Now the matrix of $I_\Lambda(\lambda)$ with respect to the basis (v_k) is upper triangular with (k,k)-entry $I_{\psi_k}(\lambda) > 0$. Therefore we can give a meaning to $I_\Lambda(\lambda)^{1/2}$. Put $H_\lambda = L^2(E_\lambda)$ and let $B_2(H_\lambda)$ be the Hilbert space of Hilbert-Schmidt operators on H_λ. Let us put

$$A(H_\lambda) = \{T \in B_2(H_\lambda)\ ;\ \text{Range } T \subset V_\lambda\}.$$

It is evident that $A(H_\lambda)$ is a closed subspace of $B_2(H_\lambda)$ and so $A(H_\lambda)$ itself is a Hilbert space. Consider the Hilbert space tensor product $A(H_\lambda) \otimes E$ of two Hilbert spaces $A(H_\lambda)$ and E. This tensor product space $A(H_\lambda) \otimes E$ is regarded as the Hilbert space $B_2(E, A(H_\lambda))$ of anti-linear Hilbert-Schmidt operators mapping E to $A(H_\lambda)$ via $(T \otimes v)(u) = (v, u)T$. For $\lambda \in \Omega^*$, we define an operator $M_\Lambda(\lambda)$ on $A(H_\lambda) \otimes E$ by

$$M_\Lambda(\lambda)(T \otimes v) = T \otimes I_\Lambda(\lambda)^{1/2} v.$$

We are now in a position to define a Hilbert space $H(\Lambda)$: it consists of measurable functions Ψ on Ξ taking their value at $\lambda \in \Xi$ in $A(\mathbb{H}_\lambda) \otimes E$ such that

(i) $\Psi(\lambda) = 0$ if $\lambda \notin \Omega^*$,

(ii) $\|\Psi\|^2 = c \int_{\Omega^*} \|M_\Lambda(\lambda)\Psi(\lambda)\|^2 \rho(\lambda)\, d\lambda < \infty.$

Put $H_j(\Lambda) = \{\Psi \in H(\Lambda)\ ;\ \Psi(\lambda) \in A(\mathbb{H}_\lambda) \otimes (E_1 \oplus \dots \oplus E_j)\}$ and

$$T_j\Psi(\lambda) = \Psi(\lambda)v_j \in A(\mathbb{H}_\lambda) \qquad (\Psi \in H_j(\Lambda)).$$

T_j is a bounded operator $H_j(\Lambda) \to \mathcal{H}^2(\Omega^*, \psi_j)$ and its range is dense. Let $H^1(\Lambda) = H_1(\Lambda)$ and $H^j(\Lambda) =$ the orthogonal complement of $H_{j-1}(\Lambda)$ in $H_j(\Lambda)$ $(j = 2, \dots, d)$. Then, $H^j(\Lambda)$ is unitarily isomorphic to $\mathcal{H}^2(\Omega^*, \psi_j)$ by U_j, where U_j is the partial isometry appearing in the polar decomposition of the operator T_j. Therefore we have an orthogonal decomposition $H(\Lambda) = \bigoplus_{j=1}^{d} H^j(\Lambda)$.

Theorem 2. $\mathcal{H}(\Lambda)$ is unitarily isomorphic to $H(\Lambda)$ via the following diagram:

$$\begin{tikzcd}[column sep=large]
\bigoplus_{j=1}^{d} \mathcal{H}^j(\Lambda) = \mathcal{H}(\Lambda) \arrow[r, dashed, "\simeq"] & H(\Lambda) = \bigoplus_{j=1}^{d} H^j(\Lambda) \\
\text{(for each } j) \quad \mathcal{H}^j(\Lambda) \arrow[u, phantom] \arrow[d, "\wr\|\ U_j"'] & H^j(\Lambda) \arrow[d, "\wr\|\ U_j"] \\
H^2(D, \psi_j) \arrow[r, "\simeq", "\text{(Theorem 1)}"'] & \mathcal{H}^2(\Omega^*, \psi_j)
\end{tikzcd}$$

References

[1] Harish-Chandra, Representations of semisimple Lie groups IV, V, VI, Amer. J. Math., 77 (1955), 743-777; Ibid., 78 (1956), 1-41: Ibid., 78 564-628.

[2] C. C. Moore, Compactification of symmetric spaces II: the Cartan domains, Amer. J. Math., 86 (1964), 358-378.

[3] R. D. Ogden and S. Vági, Harmonic analysis of a nilpotent group and function theory on Siegel domains of type II, Adv. Math., 33 (1979), 31-92.

[4] H. Rossi and M. Vergne, Representations of certain solvable Lie groups on Hilbert spaces of holomorphic functions and the application to the holomorphic discrete series of a semisimple Lie group, J. Funct. Anal., 13 (1973), 324-389.

[5] M. Vergne and H. Rossi, Analytic continuation of holomorphic discrete series of a semisimple Lie group, Acta. Math., 136 (1976), 1-59.